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Parametric operations between 3-dimensional triangular fuzzy number and trapezoidal fuzzy set*

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Abstract. The parametric operations on $\mathbb{R}^2$ are the generalization of Zadeh's extended algebraic operations on $\mathbb{R}$. We calculated the parametric operations for two 2-dimensional generalized triangular fuzzy sets, for two 2-dimensional quadratic fuzzy numbers, and for two 2-dimensional trapezoidal fuzzy sets. We also calculated the parametric operations between 2-dimensional triangular fuzzy number and 2-dimensional trapezoidal fuzzy sets.

The parametric operations on $\mathbb{R}^3$ are the generalization of Zadeh's extended algebraic operations on $\mathbb{R}^2$, and thus become the generalization of Zadeh's extended algebraic operations on $\mathbb{R}$.

In this paper, we calculate the parametric operations between 3-dimensional triangular fuzzy number and trapezoidal fuzzy set.

AMS Subject Classification (2020): 47N99

Keywords: parametric operation, 3-dimensional triangular fuzzy number, 3-dimensional trapezoidal fuzzy set

1. Introduction

The results for Zadeh's extended algebraic operations on $\mathbb{R}$ have been utilized in principle operations for two fuzzt sets and fuzzy logics. The parametric operations on $\mathbb{R}^2$ are the generalization of Zadeh's extended algebraic operations on $\mathbb{R}$([10]-[12]). We generalized triangular, quadratic, and trapezoidal fuzzy sets from $\mathbb{R}$ to $\mathbb{R}^2$([1]). We generalized the algebraic

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operations for triangular, quadratic, and trapezoidal fuzzy sets on $\mathbb{R}$ to parametric operations for triangular, quadratic, and trapezoidal fuzzy sets on $\mathbb{R}^2$, respectively([2]-[6]). We generalized the triangular fuzzy numbers from $\mathbb{R}^2$ to $\mathbb{R}^3$([7]).

By defining a parametric operator between two α-cuts, we defined a parametric operator for two triangular fuzzy numbers defined on $\mathbb{R}^3$. We calculated Zadeh's max-min composition operator for 3-dimensional triangular fuzzy number([7]). And we presented the calculation in 3-dimensional graphs for generalized triangular fuzzy sets([8],[9]).

In this paper, we generalize the trapezoidal fuzzy set from $\mathbb{R}^2$ to $\mathbb{R}^3$, and calculate the parametric operations between 3-dimensional triangular fuzzy number and trapezoidal fuzzy set.

2. Parametric operations for two triangular fuzzy numbers defined on $\mathbb{R}^3$

We define the 3-dimensional triangular fuzzy numbers on $\mathbb{R}^3$ as a generalization of triangular fuzzy numbers on $\mathbb{R}^2$.

Definition 2.1 [7]. A fuzzy set A with a membership function

$$\mu_A(x,y,z) = \begin{cases} 1-\sqrt{\dfrac{(x-x_1)^2}{a^2}+\dfrac{(y-y_1)^2}{b^2}+\dfrac{(z-z_1)^2}{c^2}}, \\ \qquad \text{if} \quad b^2c^2(x-x_1)^2+c^2a^2(y-y_1)^2+a^2b^2(z-z_1)^2 \le a^2b^2c^2, \\ 0, \quad \text{otherwise}, \end{cases}$$

where $a,b,c>0$ is called the *3-dimensional triangular fuzzy number* and denoted by $(a,\ x_1,\ b,\ y_1,\ c,\ z_1)^3$.

Note that $\mu_A(x,y)$ is a cone in $\mathbb{R}^2$, but it is not possible to know the shape of $\mu_A(x,y,z)$ in $\mathbb{R}^3$. We define the parametric operations between two 3-

dimensional triangular fuzzy numbers. To achieve this, we have to calculate operations between α-cuts in $\mathbb{R}^3$.

The α-cuts are regions in $\mathbb{R}^2$, but in $\mathbb{R}^3$, the α-cuts are some subsets of $\mathbb{R}^3$, which makes the existing method of calculations between α-cuts inapplicable. We interpret the existing method from a different perspective, and apply the method to the subset valued α-cuts on $\mathbb{R}^3$. The α-cut A_α of a 3-dimensional triangular fuzzy number $A = (a, x_1, b, y_1, c, z_1)^3$ is the following set

$$A_\alpha = \left\{(x,y,z) \in \mathbb{R}^3 \middle| \left(\frac{x - x_1}{a(1-\alpha)}\right)^2 + \left(\frac{y - y_1}{b(1-\alpha)}\right)^2 + \left(\frac{z - z_1}{c(1-\alpha)}\right)^2 \leq 1\right\}.$$

Definition 2.2 [7]. A 3-dimensional fuzzy number A defined on $\mathbb{R}^3$ is called *convex* fuzzy number if for all $\alpha \in (0,1)$, the α-cuts $A_\alpha = \{(x,y,z) \in \mathbb{R}^3 | \mu_A(x,y,z) \geq \alpha\}$ are convex subsets in $\mathbb{R}^3$.

Theorem 2.3 [7]. *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^3$ and $A^\alpha = \{(x,y,z) \in \mathbb{R}^3 | \mu_A(x,y,z) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(s), f_2^\alpha(s,t)$ and $f_3^\alpha(s,t)\left(0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\right)$ such that*

$$A^\alpha = \left\{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\right\}.$$

Definition 2.4 [7]. Let A and B be two continuous convex fuzzy numbers defined on $\mathbb{R}^3$, and

$$A^\alpha = \left\{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\right\},$$
$$B^\alpha = \left\{(g_1^\alpha(s), g_2^\alpha(s,t), g_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\right\}$$

be the α-sets of A and B, respectively.

For $\alpha \in (0,1)$, we define that the parametric addition, parametric subtraction, parametric multiplication and parametric division of two fuzzy numbers A and B are fuzzy numbers that have their α-sets as follows:

(1) parametric addition $A(+)_pB$:

$$(A(+)_pB)^\alpha = \Big\{(f_1^\alpha(s) + g_1^\alpha(s),\ f_2^\alpha(s,t) + g_2^\alpha(s,t), f_3^\alpha(s,t) + g_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\Big\}$$

(2) parametric subtraction $A(-)_pB$:

$$(A(-)_pB)^\alpha = \Big\{(f_1^\alpha(s) - g_1^\alpha(s+\pi),\ f_2^\alpha(s,t) - g_2^\alpha(s+\pi,t), f_3^\alpha(s,t) - g_3^\alpha(s+\pi,t)) \in \mathbb{R}^3 | 0 \le s \le \pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\Big\},$$

$$(A(-)_pB)^\alpha = \Big\{(f_1^\alpha(s) - g_1^\alpha(s-\pi),\ f_2^\alpha(s,t) - g_2^\alpha(s-\pi,t), f_3^\alpha(s,t) - g_3^\alpha(s-\pi,t)) \in \mathbb{R}^3 | \pi \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\Big\}$$

(3) parametric multiplication $A(\cdot)_pB$:

$$(A(\cdot)_pB)^\alpha = \Big\{(f_1^\alpha(s) \cdot g_1^\alpha(s),\ \ f_2^\alpha(s,t) \cdot g_2^\alpha(s,t), f_3^\alpha(s,t) \cdot g_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\Big\}$$

(4) parametric division $A(/)_pB$:

$$(A(/)_pB)^\alpha = \left\{\left(\frac{f_1^\alpha(s)}{g_1^\alpha(s+\pi)}, \frac{f_2^\alpha(s,t)}{g_2^\alpha(s+\pi,t)}, \frac{f_3^\alpha(s,t)}{g_3^\alpha(s+\pi,t)}\right) \in \mathbb{R}^3 | 0 \le s \le \pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\right\},$$

$$(A(/)_pB)^\alpha = \left\{\left(\frac{f_1^\alpha(s)}{g_1^\alpha(s-\pi)}, \frac{f_2^\alpha(s,t)}{g_2^\alpha(s-\pi,t)}, \frac{f_3^\alpha(s,t)}{g_3^\alpha(s-\pi,t)}\right) \in \mathbb{R}^3 | \pi \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\right\}$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim\limits_{\alpha\to 0^+}(A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim\limits_{\alpha\to 1^-}(A(*)_pB)^\alpha$, where $* = +,\ -,\ \cdot,\ /$.

Theorem 2.5 [7] *Let $A = (a_1,\ x_1,\ b_1,\ y_1,\ c_1,\ z_1)^3$ and $B = (a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2)^3$ be two 3-dimensional triangular fuzzy numbers. Then, we have the following:*

(1)

$$A(+)_pB = \Big(a_1 + a_2,\ x_1 + x_2,\ b_1 + b_2,\ y_1 + y_2,\ c_1 + c_2,\ z_1 + z_2\Big)^3.$$

(2)

$$A(-)_pB = \Big(a_1 + a_2,\ x_1 - x_2,\ b_1 + b_2,\ y_1 - y_2,\ c_1 + c_2,\ z_1 - z_2\Big)^3.$$

(3)

$$(A(\cdot)_pB)^\alpha = \Big\{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\Big\},$$

where

$$\begin{aligned} x_\alpha(s) &= x_1x_2 + (x_1a_2 + x_2a_1)(1-\alpha)\cos s + a_1a_2(1-\alpha)^2\cos^2 s, \\ y_\alpha(s,t) &= y_1y_2 + (y_1b_2 + y_2b_1)(1-\alpha)\sin s\cos t + b_1b_2(1-\alpha)^2 \\ &\quad \sin^2 s\cos^2 t, \\ z_\alpha(s,t) &= z_1z_2 + (z_1c_2 + z_2c_1)(1-\alpha)\sin s\sin t + c_1c_2(1-\alpha)^2 \\ &\quad \sin^2 s\sin^2 t. \end{aligned}$$

(4)

$$(A(/)_pB)^\alpha = \Big\{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\Big\},$$

where

$$x_\alpha(s) = \frac{x_1 + a_1(1-\alpha)\cos s}{x_2 - a_2(1-\alpha)\cos s},$$
$$y_\alpha(s,t) = \frac{y_1 + b_1(1-\alpha)\sin s\cos t}{y_2 - b_2(1-\alpha)\sin s\cos t},$$
$$z_\alpha(s,t) = \frac{z_1 + c_1(1-\alpha)\sin s\sin t}{z_2 - c_2(1-\alpha)\sin s\sin t}.$$

Therefore, $A(+)_pB$ and $A(-)_pB$ become 3-dimensional triangular fuzzy numbers, but $A(\cdot)_pB$ and $A(/)_pB$ are not 3-dimensional triangular fuzzy numbers.

Example 2.6 [7]. Let $A = (6,\ 3,\ 8,\ 5,\ 4,\ 7)^3$ and $B = (4,\ 2,\ 5,\ 3,\ 6,\ 4)^3$. Then, by Theorem 2.5, we have the following:

(1) $A(+)_pB = (10,\ 5,\ 13,\ 8,\ 10,\ 11)^3$

(2) $A(-)_pB = (10,\ 1,\ 13,\ 2,\ 10,\ 3)^3$

(3) $(A(\cdot)_pB)^\alpha = \left\{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\right\}$, where

$$x_\alpha(s) = 6 + 24(1-\alpha)\cos s + 24(1-\alpha)^2\cos^2 s,$$
$$y_\alpha(s,t) = 15 + 49(1-\alpha)\sin s\cos t + 40(1-\alpha)^2\sin^2 s\cos^2 t,$$
$$z_\alpha(s,t) = 28 + 58(1-\alpha)\sin s\sin t + 24(1-\alpha)^2\sin^2 s\sin^2 t.$$

(4)

$$(A(/)_pB)^\alpha = \left\{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\right\},$$

where

$$x_\alpha(s) = \frac{3+6(1-\alpha)\cos s}{2-4(1-\alpha)\cos s},$$
$$y_\alpha(s,t) = \frac{5+8(1-\alpha)\sin s\cos t}{3-5(1-\alpha)\sin s\cos t},$$
$$z_\alpha(s,t) = \frac{7+4(1-\alpha)\sin s\sin t}{4-6(1-\alpha)\sin s\sin t}.$$

Therefore, $A(+)_pB$ and $A(-)_pB$ become 3-dimensional triangular fuzzy numbers, but $A(\cdot)_pB$ and $A(/)_pB$ are not 3-dimensional triangular fuzzy numbers.

3. Parametric operations between 3-dimensional triangular fuzzy number and trapezoidal fuzzy set

In this section, we define the 3-dimensional trapezoidal fuzzy sets on $\mathbb{R}^3$ as a generalization of trapezoidal fuzzy sets on $\mathbb{R}^2$. We then calculate the parametric operations between 3-dimensional triangular fuzzy number and trapezoidal fuzzy set on $\mathbb{R}^3$.

Definition 3.1. A fuzzy set B with a membership function

$$\mu_B(x,y,z) = \begin{cases} h - \sqrt{\dfrac{(x-x_2)^2}{a_2^2} + \dfrac{(y-y_2)^2}{b_2^2} + \dfrac{(z-z_2)^2}{c_2^2}}, \\ \qquad \text{if} \quad h-1 \le \sqrt{\dfrac{(x-x_2)^2}{a_2^2} + \dfrac{(y-y_2)^2}{b_2^2} + \dfrac{(z-z_2)^2}{c_2^2}} \le h, \\ 1, \quad \text{if} \quad 0 \le \sqrt{\dfrac{(x-x_2)^2}{a_2^2} + \dfrac{(y-y_2)^2}{b_2^2} + \dfrac{(z-z_2)^2}{c_2^2}} \le h-1, \\ 0, \quad \text{otherwise}, \end{cases}$$

where $a_2, b_2, c_2 > 0$ and $1 < h$ is called the *3-dimensional trapezoidal fuzzy set* and denoted by $B = ((h,\ a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2))^3$.

The α-cut B_α of a 3-dimensional trapezoidal fuzzy set $B = ((h,\ a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2))^3$ is the following set

$$B_\alpha = \Big\{(x, y, z) \in \mathbb{R}^3 \Big| \Big(\frac{x - x_2}{a_2(h-\alpha)}\Big)^2 + \Big(\frac{y - y_2}{b_2(h-\alpha)}\Big)^2 + \Big(\frac{z - z_2}{c_2(h-\alpha)}\Big)^2 \le 1\Big\}.$$

Note that if $0 < h < 1$, $((h,\ a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2))^3$ becomes a generalized 3-dimensional triangular fuzzy set and if $1 < h$, $((h,\ a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2))^3$ becomes a 3-dimensional trapezoidal fuzzy set.

Theorem 3.2. *Let $A = (a_1,\ x_1,\ b_1,\ y_1,\ c_1,\ z_1)^3$ be a 3-dimensional triangular fuzzy number and $B = ((h, a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2))^3$ be a 3-dimensional trapezoidal fuzzy set. Then we have the following:*

(1) *For $0 < \alpha < 1$, the α-set $(A(+)_pB)^\alpha$ of $A(+)_pB$ is*

$$\Big\{(x, y, z) \in \mathbb{R}^3 \Big| \Big(\frac{x - x_1 - x_2}{a_1(1-\alpha) + a_2(h-\alpha)}\Big)^2 + \Big(\frac{y - y_1 - y_2}{b_1(1-\alpha) + b_2(h-\alpha)}\Big)^2 + \Big(\frac{z - z_1 - z_2}{c_1(1-\alpha) + c_2(h-\alpha)}\Big)^2 = 1\Big\}.$$

(2) *For $0 < \alpha < 1$, the α-set $(A(-)_pB)^\alpha$ of $A(-)_pB$ is*

$$\Big\{(x, y, z) \in \mathbb{R}^3 \Big| \Big(\frac{x - x_1 + x_2}{a_1(1-\alpha) + a_2(h-\alpha)}\Big)^2 + \Big(\frac{y - y_1 + y_2}{b_1(1-\alpha) + b_2(h-\alpha)}\Big)^2 + \Big(\frac{z - z_1 + z_2}{c_1(1-\alpha) + c_2(h-\alpha)}\Big)^2 = 1\Big\}.$$

(3) *For $0 < \alpha < 1$, $(A(\cdot)_pB)^\alpha = \Big\{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}\Big\}$, where*

$$\begin{aligned} x_\alpha(s) &= x_1x_2 + (x_1a_2(h-\alpha) + x_2a_1(1-\alpha))\cos s \\ &\quad + a_1a_2(h-\alpha)(1-\alpha)\cos^2 s, \\ y_\alpha(s,t) &= y_1y_2 + (y_1b_2(h-\alpha) + y_2b_1(1-\alpha))\sin s\cos t \\ &\quad + b_1b_2(h-\alpha)(1-\alpha)\sin^2 s\cos^2 t, \\ z_\alpha(s,t) &= z_1z_2 + (z_1c_2(h-\alpha) + z_2c_1(1-\alpha))\sin s\sin t \\ &\quad + c_1c_2(h-\alpha)(1-\alpha)\sin^2 s\sin^2 t. \end{aligned}$$

(4) *For* $0 < \alpha < 1$, $(A(/)_p B)^\alpha = \left\{ (x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}$, *where*

$$x_\alpha(s) = \frac{x_1 + a_1(1-\alpha)\cos s}{x_2 - a_2(h-\alpha)\cos s},$$
$$y_\alpha(s,t) = \frac{y_1 + b_1(1-\alpha)\sin s \cos t}{y_2 - b_2(h-\alpha)\sin s \cos t},$$
$$z_\alpha(s,t) = \frac{z_1 + c_1(1-\alpha)\sin s \sin t}{z_2 - c_2(h-\alpha)\sin s \sin t}.$$

Proof. Since A and B are continuous convex fuzzy numbers defined on $\mathbb{R}^3$, by Theorem 2.3, there exists $f_1^\alpha(s), g_1^\alpha(s), f_i^\alpha(s,t), g_i^\alpha(s,t)$ $(i = 2, 3)$ such that

$$A^\alpha = \left\{ (f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\},$$
$$B^\alpha = \left\{ (g_1^\alpha(s), g_2^\alpha(s,t), g_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}.$$

Since $A = (a_1,\ x_1,\ b_1,\ y_1,\ c_1,\ z_1)^3$ and $B = ((h, a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2))^3$, we have

$$f_1^\alpha(s) = x_1 + a_1(1-\alpha)\cos s, \quad f_2^\alpha(s,t) = y_1 + b_1(1-\alpha)\sin s \cos t,$$
$$f_3^\alpha(s,t) = z_1 + c_1(1-\alpha)\sin s \sin t,$$

and

$$g_1^\alpha(s) = x_2 + a_2(h-\alpha)\cos s, \quad g_2^\alpha(s,t) = y_2 + b_2(h-\alpha)\sin s \cos t,$$
$$g_3^\alpha(s,t) = z_2 + c_2(h-\alpha)\sin s \sin t.$$

(1) Since

$$f_1^\alpha(s) + g_1^\alpha(s) = x_1 + x_2 + (a_1(1-\alpha) + a_2(h-\alpha))\cos s,$$
$$f_2^\alpha(s,t) + g_2^\alpha(s,t) = y_1 + y_2 + (b_1(1-\alpha) + b_2(h-\alpha))\sin s \cos t,$$
$$f_3^\alpha(s,t) + g_3^\alpha(s,t) = z_1 + z_2 + (c_1(1-\alpha) + c_2(h-\alpha))\sin s \sin t,$$

we have

$$(A(+)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| \Big(\frac{x-x_1-x_2}{a_1(1-\alpha)+a_2(h-\alpha)}\Big)^2 + \Big(\frac{y-y_1-y_2}{b_1(1-\alpha)+b_2(h-\alpha)}\Big)^2 + \Big(\frac{z-z_1-z_2}{c_1(1-\alpha)+c_2(h-\alpha)}\Big)^2 = 1\Big\}.$$

(2) If $0 \leq s \leq \pi$, we have

$$\begin{aligned} f_1^\alpha(s) - g_1^\alpha(s+\pi) &= x_1 - x_2 + (a_1(1-\alpha) - a_2(h-\alpha))\cos s \\ f_2^\alpha(s,t) - g_2^\alpha(s+\pi,t) &= y_1 - y_2 + (b_1(1-\alpha) - b_2(h-\alpha))\sin s \cos t \\ f_3^\alpha(s,t) - g_3^\alpha(s+\pi,t) &= z_1 - z_2 + (c_1(1-\alpha) - c_2(h-\alpha))\sin s \sin t. \end{aligned}$$

In the case of $\pi \leq s \leq 2\pi$, we have

$$\begin{aligned} f_1^\alpha(s) - g_1^\alpha(s-\pi) &= f_1^\alpha(s) - g_1^\alpha(s+\pi), \\ f_2^\alpha(s,t) - g_2^\alpha(s-\pi,t) &= f_2^\alpha(s,t) - g_2^\alpha(s+\pi,t) \\ f_3^\alpha(s,t) - g_3^\alpha(s-\pi,t) &= f_3^\alpha(s,t) - g_3^\alpha(s+\pi,t). \end{aligned}$$

Thus

$$(A(-)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| \Big(\frac{x-x_1+x_2}{a_1(1-\alpha)+a_2(h-\alpha)}\Big)^2 + \Big(\frac{y-y_1+y_2}{b_1(1-\alpha)+b_2(h-\alpha)}\Big)^2 + \Big(\frac{z-z_1+z_2}{c_1(1-\alpha)+c_2(h-\alpha)}\Big)^2 = 1\Big\}.$$

(3) Let $(A(\cdot)_pB)^\alpha = \Big\{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\Big\}$, then

$$
\begin{aligned}
x_\alpha(s) &= x_1x_2 + (x_1a_2(h-\alpha) + x_2a_1(1-\alpha))\cos s \\
&\quad + a_1a_2(h-\alpha)(1-\alpha)\cos^2 s, \\
y_\alpha(s,t) &= y_1y_2 + (y_1b_2(h-\alpha) + y_2b_1(1-\alpha))\sin s\cos t \\
&\quad + b_1b_2(h-\alpha)(1-\alpha)\sin^2 s\cos^2 t, \\
z_\alpha(s,t) &= z_1z_2 + (z_1c_2(h-\alpha) + z_2c_1(1-\alpha))\sin s\sin t \\
&\quad + c_1c_2(h-\alpha)(1-\alpha)\sin^2 s\sin^2 t.
\end{aligned}
$$

(4) Let $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$, then

$$
\begin{aligned}
x_\alpha(s) &= \frac{x_1 + a_1(1-\alpha)\cos s}{x_2 - a_2(h-\alpha)\cos s}, \\
y_\alpha(s,t) &= \frac{y_1 + b_1(1-\alpha)\sin s\cos t}{y_2 - b_2(h-\alpha)\sin s\cos t} \\
z_\alpha(s,t) &= \frac{z_1 + c_1(1-\alpha)\sin s\sin t}{z_2 - c_2(h-\alpha)\sin s\sin t}.
\end{aligned}
$$

The proof is complete. $\square$

Example 3.3. Let $A = (6,\ 3,\ 8,\ 5,\ 4,\ 7)^3$ and $B = ((3,\ 4,\ 2,\ 5,\ 3,\ 6,\ 4))^3$. Then by Theorem 3.2, we have the following:

(1) For $0 < \alpha < 1$, the α-set of $A(+)_pB$ is

$$
\begin{aligned}
(A(+)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| &\Big(\frac{x-5}{6(1-\alpha)+4(3-\alpha)}\Big)^2 \\
&+ \Big(\frac{y-8}{8(1-\alpha)+5(3-\alpha)}\Big)^2 + \Big(\frac{z-11}{4(1-\alpha)+6(3-\alpha)}\Big)^2 \\
&= 1\Big\}.
\end{aligned}
$$

(2) For $0 < \alpha < 1$, the α-set of $A(-)_pB$ is

$$
\begin{aligned}
(A(-)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| &\Big(\frac{x-1}{6(1-\alpha)+4(3-\alpha)}\Big)^2 \\
&+ \Big(\frac{y-2}{8(1-\alpha)+5(3-\alpha)}\Big)^2 + \Big(\frac{z-3}{4(1-\alpha)+6(3-\alpha)}\Big)^2 = 1\Big\}.
\end{aligned}
$$

(3) For $0<\alpha<1$, $(A(\cdot)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$, where

$$x_\alpha(s) = 6 + (12(3-\alpha) + 12(1-\alpha))\cos s + 24(3-\alpha)(1-\alpha)\cos^2 s,$$

$$\begin{aligned} y_\alpha(s,t) = 15 &+ (25(3-\alpha) + 24(1-\alpha))\sin s \cos t \\ &+ 40(3-\alpha)(1-\alpha)\sin^2 s \cos^2 t, \\ z_\alpha(s,t) = 28 &+ (42(3-\alpha) + 16(1-\alpha))\sin s \sin t \\ &+ 24(3-\alpha)(1-\alpha)\sin^2 s \sin^2 t. \end{aligned}$$

(4) For $0<\alpha<1$, $(A(/)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$, where

$$\begin{aligned} x_\alpha(s) &= \frac{3+6(1-\alpha)\cos s}{2-4(3-\alpha)\cos s}, \\ y_\alpha(s,t) &= \frac{5+8(1-\alpha)\sin s \cos t}{3-5(3-\alpha)\sin s \cos t}, \\ z_\alpha(s,t) &= \frac{7+4(1-\alpha)\sin s \sin t}{4-6(3-\alpha)\sin s \sin t}. \end{aligned}$$

We give the membership functions of $\mu_{A(+)B}(x)$, $\mu_{A(-)B}(x)$, $\mu_{A(\cdot)B}(x)$ and $\mu_{A(/)B}(x)$ with their α-sets. Thus we do not know the concrete form of the membership functions. We proved that the functions f_i and $g_i (i = +, -, \cdot, /)$ are in one to one correspondence. Thus we know the unique existence of membership functions.

In the following, we get the graphs of membership functions using Mathematica. The value of the membership function is expressed with color density since a 3-dimensional graph can not be drawn. The value of the membership function for each point on the cut plane is also expressed with color density.

A := ImplicitRegion[0≤Sqrt[(x-3)^2/6+(y-5)^2/8+(z-7)^2/4] ≤1, {x,y,z}];
DensityPlot3D[1-Sqrt[(x-3)^2/6+(y-5)^2/8+(z-7)^2/4], {x,y,z} reg1,PlotPoints

→100,ColorFunction→"SunsetColors",OpacityFunction→0.05,BoxRatios →{Sqrt[6],Sqrt[8],Sqrt[4]},PlotLegends→Automatic]

B := reg2=ImplicitRegion[0≤Sqrt[(x-2)^2/4+(y-3)^2/5+(z-4)^2/6]≤3,{x,y,z}]; DensityPlot3D[3-Sqrt[(x-2)^2/4+(y-3)^2/5+(z-4)^2/6],{x,y,z}reg2,PlotPoints →100,ColorFunction→"SunsetColors",OpacityFunction→0.05,BoxRatios→ {Sqrt[4],Sqrt[5],Sqrt[6]},PlotLegends→Automatic]

A+B := ContourPlot3D[((x-5)/(18-10a))^2+((y-8)/(23-13a))^2+((z-11)/(22 -10a))^2==1,{x,-15,25},{y,-21,38},{z,-11,41},ContourStyle→Directive [RGB Color[1-a,1-a,1-a],Opacity[0.1]],Mesh→None,PlotPoints→100, BoxRatios→ {1,1,0.5},BoundaryStyle→None];tg=Table[g[i],{i,0,1,0.05}];Show[tg]

A-B := ContourPlot3D[((x-1)/(18-10a))^2+((y-2)/(23-13a))^2+((z-3)/ (22- 10a))^2==1,{x,-34,40},{y,-34,40},{z,-34,40},ContourStyle→Directiv [RGB Color[1-a,1-a,1-a],Opacity[0.1]],Mesh→None,PlotPoints→100, BoxRatios→ {1,1,0.5},BoundaryStyle→None];tg=Table[g1[i],i,0,1,0.1];Show[tg]

A·B := ParametricPlot3D[{6+(48-24a)Cos[s]+24(3-a)(1-a)(Cos[s])^2,15+(99 -49a)Sin[s]Cos[t]+40(3-a)(1-a)(Sin[s])^2(Cos[t])^2,28+(142-58a)Sin[s]Sin[t] + 24(3-a)(1-a)(Sin[s])^2(Sin[t])^2},{s,0,2Pi},{t,-Pi/2,Pi/2},PlotStyle→ Directive [RGB Color[0.2,0.5+a/2,0.5+a/2],Opacity[0.3]],BoxRatios→{1,1,1}];tg= Table[g[i],{i,0,1,0.1}];Show[tg]

A/B := ParametricPlot3D[{(3+6(1-a)Cos[s])/(2-4(3-a)Cos[s]),(5+8(1-a) Sin[s] Cos[t])/(3-5(3-a)/4Sin[s]Cos[t]),(7+ 4(1-a)Sin[s]Sin[t])/(4-6(3-a)Sin[s] Sin[t])}, {s,0,2Pi},{t,-Pi/2,Pi/2},PlotStyle→Directive[RGBColor[0.2,0.5+ a/2,0.5+a/2], Opacity[0.3]],BoxRatios→{1,1,1}];tg=Table[g[i],{i,0,1,0.1}]; Show[tg]

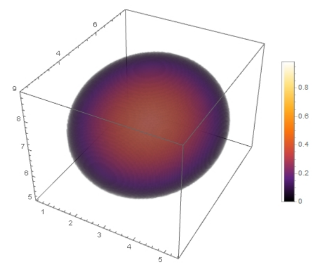

Fig. 1: $\mu_A(x)$

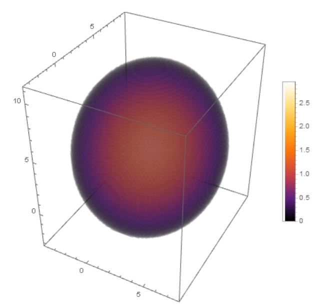

Fig. 2: $\mu_B(x)$

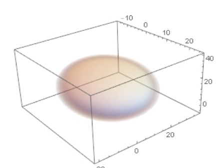

Fig. 3: $\mu_{A+B}(x)$

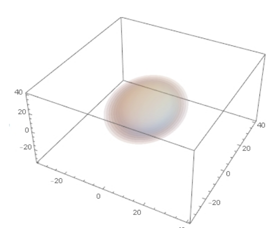

Fig. 4: $\mu_{A-B}(x)$

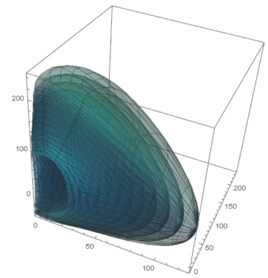

Fig. 5: $\mu_{A\cdot B}(x)$

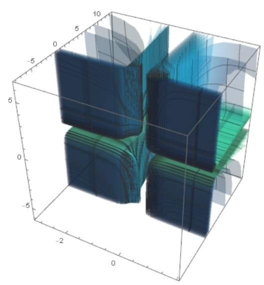

Fig. 6: $\mu_{A/B}(x)$

4. Conclusion

We have calculated the parametric operations between 3-dimensional triangular fuzzy number and trapezoidal fuzzy set. The results of this paper have been illustrated with the help of an Example 3.3. For 3-dimensional triangular fuzzy number A and 3-dimensional trapezoidal fuzzy sets B, $A(+)B$ and $A(-)B$ became 3-dimensional trapezoidal fuzzy sets([Figure 5, Figure 6]).

The form of $A(\cdot)B$ seems to be usable for application if it is modified a little, but the form of $A(/)B$ is so broken that it can not be used at all in applications. However, Of course, since $A(+)B$ and $A(-)B$ are smoothly expressed, they can be applied in many areas without modification.

The next research can be extended to finite dimensional case. And thus this study will be of great help in the study of extension to finite dimensional case.

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